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Modifications on Ranking Functions

Abstract

Ranking theory, firstly presented in Spohn (1988) and further developed by Spohn (1999), Huber (2006, 2007), Spohn (2009, 2012), Huber (2012), Skovgaard-Olsen (2016), Raidl and Skovgaard-Olsen (2017), and Huber (2020), is a formal epistemology that represents epistemic states in terms of ranking functions, and rational changes of epistemic states in terms of ranking conditionalizations. Generally speaking, propositions are considered to be the objects of belief and a ranking function k is used to coordinate the relations between propositions and numbers. Since ranking theory develops through many works, I choose Spohn (2012) as my main reference to ranking theory, because Spohn presents a complete summary of ranking theory in this book. The definitions concerning ranking functions and ranking coordinalizations in Spohn (2012) seem to be somehow ambiguous, though. My aim in this article is to show the ambiguity concerning the definitions of ranking functions and ranking conditionalizations, and to make some alternative definitions to clarify them, and to make them more perspicuous.

Keywords: Ranking functions; Ranking conditionalizations; Ranking theory

1 Introduction

Ranking theory was originally put forth in Spohn (1988), and further developed by Spohn (1999), Huber (2006, 2007), Spohn (2009, 2012), Huber (2012), Skovgaard-Olsen (2016), Raidl and Skovgaard-Olsen (2017), and Huber (2020). Halpern (2017, p.294) comments that "specifically, possibility measures, ranking functions, and plausibility measures – provide good frameworks for capturing both default reasoning and counterfactual reasoning."

Generally speaking, ranking theory is a formal epistemology that regards the objects of belief as propositions, denoted by capital letters $A, B, \ldots$. A ranking theory contains the following elements: a space of possibilities W, an algebra $\mathscr{A}$ over W, a ranking function $k(\cdot)$ over W and $\mathscr{A}$, and different ranking conditionalizations. The propositions are considered as sets of possibilities in $\mathscr{A}$. A ranking function k coordinates each possibility w with a number, while it also coordinates each proposition with a number. Since a ranking function k coordinates both possibilities and propositions with numbers, a ranking function k is ambiguous, for the following reasons:

(i) Observe that the domain of a ranking function k is not defined clearly, it can be $W \cup \mathscr{A}$. If it is the case, several questions can be proposed concerning the domain of k. For instance, is $W \cup \mathscr{A}$ an algebra? If this is the case, $W \cup \mathscr{A}$ may contain a non-proposition, like $\{w, A\}$. If it is not the case, what can be a proper combination of W and $\mathscr{A}$?

(ii) Suppose the domain of a ranking function k includes both W and $\mathscr{A}$. Let $k^{-1}(n) \neq \varnothing$ corresponds to $X \subseteq W$, and $\mathcal{B} \subseteq \mathscr{A}$. It gives rise to a question concerning under which condition it is the case that $X \in \mathscr{A}$ and $X \in \mathcal{B}$.

Given the fact that the ambiguity contained in the current definitions gives rise to these questions, I suggest we can make some modifications in the following way to improve and clarify these definitions:

(i) to consider two ranking functions, $k : W \rightarrow N^+$ and $r : \mathscr{A} \rightarrow N^+$, as alternatives to the original definition of a ranking function k;

(ii) to define a ranking pair $< k, r >$ to show the dependence of ranking functions, and to define a ranking structure $<< W, \mathscr{A} >, < k, r >>$ to represent the connections between W, $\mathscr{A} >$, k, and r;

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(iii) to consider two conditional ranking functions, $ck :< W, \mathscr{A} > \to N^+$ and $cr :< \mathscr{A}, \mathscr{A} > \to N^+$, as alternatives to the original definition of a conditional ranking function $k(w \mid A)$ and $k(B \mid A)$;

(iv) to define a pair conditionalization $< ck, cr >$ to show their dependence and a pair conditionalization structure $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$ to show the connections between W, $\mathscr{A}$, k, r, ck, and cr;

(v) to consider alternative definitions of the evidence-oriented conditionalization and the result-oriented conditionalization on the basis of a pair conditionalization structure $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$.

The rest of the article is organized as follows. In section 2, I will introduce several basic definitions of the ranking theory that are quoted from Spohn (2012). In section 3, I will present modified alternative definitions. In section 4, I will return to the two questions (i) and (ii) for a discussion: I will show the convenience of using new alternatives definitions. The concluding remarks are made at the end.

2 Basic Ranking Theory

In ranking theory, a possibility space, denoted by W, is a set of possibilities, which can be finite or infinite. Propositions, denoted by A, B, C and so on, are subsets of W which correspond to the standard propositional language:

> Set-theoretically, the negation of a proposition A is represented by its complement $\bar{A}$ (relative to W), and the conjunction and disjunction of two propositions A and B is, respectively, represented by the intersection $A \cap B$ and the union $A \cup B$. (p. 17)

Spohn defines an algebra $\mathscr{A}$ over W in Spohn (2012, p. 17) to make some restrictions on the propositions discussed in ranking theory:

Definition 1. *$\mathscr{A}$ is an algebra over W iff $\mathscr{A} \subseteq \mathcal{P}(W)$, the power set of W, such that:*

(i) $W \in \mathscr{A}$,

(ii) if $A \in \mathscr{A}$, then $\bar{A} \in \mathscr{A}$,

(iii) if $A, B \in \mathscr{A}$, then $A \cup B \in \mathscr{A}$. $\mathscr{A}$ is a σ-algebra over W iff moreover

(iv) for each countable $\mathcal{B} \subseteq \mathscr{A}$, $\bigcup \mathcal{B} \in \mathscr{A}$,
and a complete algebra *over W iff moreover*

(v) for each (uncountable) $\mathcal{B} \subseteq \mathscr{A}$, $\bigcup \mathcal{B} \in \mathscr{A}$.

In this way, the objects of belief in ranking theory are the propositions that belong to an algebra A over W. According to (p.70), a ranking function is defined as follows:

Definition 2. *Let $\mathscr{A}$ be a complete algebra over W. Then k is an $\mathscr{A}$-measurable completely minimitive natural negative ranking function iff k is a function from W into $N^+ = N \cup \{\infty\}$ such that $k^{-1}(0) \neq \varnothing$ and $k^{-1}(n) \in \mathscr{A}$ for each $n \in N^+$. k is extended to propositions by defining $k(\varnothing) = \infty$ and $k(A) = min\{k(w) \mid w \in A\}$ for each nonempty $A \in \mathscr{A}$; $k(A)$ is called the negative rank of A.*

According to the definition, k is a function from W into N^+ that can be extended to $\mathscr{A}$ but it is ambiguous about how to extend the domain of k from W to a combination of W and $\mathscr{A}$.

> In (5.5) we started with a point function – ranks were first defined for possibilities in W – and then derived the set function, i.e., the ranks for propositions in $\mathscr{A}$. However, we might as well directly start with the latter. The second dimension concerns the underlying *algebra* of propositions. (5.5) assumed it to be complete, but this might be

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weakened. This corresponds to third dimension, the kind of *minimitivity* assumed. (5.5) entailed (5.8c) (complete minimitivity), but we might be content with the weaker finite minimitivity (5.8b). Finally, we might vary the range of ranking functions. Why not take real numbers as ranks instead of natural numbers as in (5.5), or ordinal numbers as previously suggested? (Spohn 2012, p. 72)

Spohn considers defining ranking function k directly from an algebra $\mathscr{A}$ to a value domain:

Definition 3. *Let A be either (a) an algebra, or (b) a σ-algebra, or (c) a complete algebra of propositions. Then k is an $\mathscr{A}$*-measurable negative ranking function *iff k is a function from $\mathscr{A}$ into either (d) $N^+ = N \cup \{\infty\}$, or (e) $R^+ = R \cup \{\infty\}$, or (f) Ω (the class of ordinals) such that*

(g) $k(W) = 0$,
(h) $k(\varnothing) = \infty$ *in cases (d) and (e) or* $k(\varnothing) = \Omega$ *in case (f), and either*
(i) $k(A \cup B) = min\{k(A), k(B)\}$ *for all* $A, B \in \mathscr{A}$, *or*
(j $k(\bigcup \mathcal{B}) = min\{k(B) \mid B \in \mathcal{B}\} forallcountable B \subseteq \mathscr{A}$, *with* $\bigcup \mathcal{B} \in \mathscr{A}$, *or*
(k) $k(\bigcup \mathcal{B}) = min\{k(B) \mid B \in \mathcal{B}\}$ *for all* $\mathcal{B} \subseteq \mathscr{A}$ *with* $\bigcup \mathcal{B} \in \mathscr{A}$.

k is called natural *in case (d),* real *in case (e),* ordinal *in case (f), (finitely) minimitive in case (i), σ-minimitive in case (j), and* completely minimitive *in case (k).*

My discussion will be restricted to natural completely minimitive ranking functions, because they are sufficient to show how a ranking theory works. I propose to define two natural completely minimitive ranking functions instead of one ranking function. That is, we can distinguish $k : W \longrightarrow N^+$ from $r : \mathscr{A} \longrightarrow N^+$.

Next, I will quote the definitions concerning conditionalizations of k by A from Spohn (2012, p. 78ff). These definitions are ambiguous, because k indicates different functions at the same time.

Definition 4. *Let k be an $\mathscr{A}$-measurable natural negative ranking function and $A \in \mathscr{A}$ with $k(A) < \infty$. Then, for any $w \in W$ the conditional negative rank of w given A is defined as*

$$k(w \mid A) = \begin{cases} k(w) - k(A) & \text{for} \quad w \in A \\ \infty & \text{for} \quad w \in \bar{A} \end{cases}$$

For any $B \in \mathscr{A}$ the conditional rank of B given A is defined as $k(B \mid A) = min\{k(w \mid A) \mid w \in B\} = k(A \cap B) - k(A)$. The function $k_A : B \to k(B \mid A)$ is called the conditionalization of k by A.

Definition 5 (Result-Oriented Conditionalization on W)**.** *Let k be a natural negative ranking function for $\mathscr{A}$ and $A \in \mathscr{A}$ such that $k(A)$, $k(\bar{A}) < \infty$, and $n \in N^+$. Then the $A \to n$-conditionalization $k_{A\to n}$ of k is defined by*

$$k_{A\to n}(w) = \begin{cases} k(w \mid A) & \text{for } w \in A, \\ k(w \mid \bar{A}) + n & \text{for } w \in \bar{A}. \end{cases}$$

The $A \to n$-conditionalization will also be called result-oriented (for reasons soon to be clear).

Definition 6 (Evidence-Oriented Conditionalization on W)**.** *Let k be a natural negative ranking function for $\mathscr{A}$, $A \in \mathscr{A}$ such that $k(A)$, $k(\bar{A}) < \infty$, and $n \in N^+$. Then the $A \uparrow n$-conditionalization $k_{A\uparrow n}$ of k is defined by*

$$k_{A\uparrow n}(w) = \begin{cases} k(w) - m & \text{for } w \in A, \\ k(w) + n - m & \text{for } w \in \bar{A}, \end{cases}$$

where $m = min\{k(A), n\}$.The $A \uparrow n$-conditionalization will also be called evidence-oriented.

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Definition 7 (Result-Oriented Conditionalization on $\mathscr{A}$)**.** *Let k be a negative ranking function for $\mathscr{A}$, $A \in \mathscr{A}$ such that $k(A), k(\bar{A}) < \infty$, and $x \in N^+$. Then the $A \to n$-conditionalization $k_{A\to n}$ of k is defined by*

$$k_{A\to n}(B) = \begin{cases} k(B \mid A) & \text{for } B \subseteq A, \\ k(B \mid \bar{A}) + x & \text{for } B \subseteq \bar{A}. \end{cases}$$

From this $k_{A\to n}(B)$ may be inferred for all other $B \in \mathscr{A}$ by the law of disjunction.

Definition 8 (Evidence-Oriented Conditionalization on $\mathscr{A}$)**.** *Let k be a negative ranking function for $\mathscr{A}$, $A \in \mathscr{A}$ such that $k(A)$, $k(\bar{A}) < \infty$, and $n \in N^+$. Then the $A \uparrow n$-conditionalization $k_{A\uparrow n}$ of k is defined by*

$$k_{A\uparrow n}(B) = \begin{cases} k(B \mid A) - m & \text{for } B \subseteq A, \\ k(B \mid \bar{A}) + n - m & \text{for } B \subseteq \bar{A}, \end{cases}$$

where $m = min\{k(A), n\}$. Again, $k_{A\uparrow n}(B)$ may be inferred for all other $B \in \mathscr{A}$ by the law of disjunction.

The definitions concerning the result-oriented and the evidence-oriented conditionalizations on $\mathscr{A}$ are ambiguous for one more reason: for a proposition C that belongs neither to A nor $\bar{A}$, C's conditional rank value is not defined explicitly.

3 New Definitions of Ranking Functions

In this section, I suggest to consider the following modified definitions.

Definition 9 (Ranking Function k)**.** *Let $\mathscr{A}$ be a complete algebra over W. Then k is an $\mathscr{A}$-measurable completely minimitive natural negative ranking function iff k is a function from W into $N^+ = N \cup \infty$ such that:*

(i) $k^{-1}(0) \neq \varnothing$,

(ii) $k^{-1}(n) \in \mathscr{A}$ *for each* $n \in N^+$.

This function is an alternative to $k : W \longrightarrow N^+$ in definition 2. According to this definition, the dependence of k on $\mathscr{A}$ can be observed clearly:

Observation 1. *A ranking function $k : W \longrightarrow N^+$ is dependent on $\mathscr{A}$ over W.*

The following definition is an alternative to $k : \mathscr{A} \longrightarrow N^+$ in definition 2.

Definition 10 (Ranking Function r)**.** *Let $\mathscr{A}$ be a complete algebra over W, k is an $\mathscr{A}$-measurable completely minimitive natural negative ranking function on W, r is a natural negative ranking function iff r is a function from $\mathscr{A}$ into $N^+ = N \cup \infty$ such that $r(\varnothing) = \infty$ and $r(A) = min\{k(w) \mid w \in A\}$ for each nonempty $A \in \mathscr{A}$; $r(A)$ is called the negative rank of A.*

This definition shows explicitly how r depends on k, which is dependent on $\mathscr{A}$. In definition 2, Spohn mentions "k is extended to propositions by defining ...". With the help of definition 9 and 10, we can explicate how k is *extended* to r:

Observation 2. *For any space of possibilities W_i and any complete algebra $\mathscr{A}_j$ over W_i, we can define $k_{j.m}$. $r_{j.m}$ is uniquely determined by W_i, $\mathscr{A}_j$, and $k_{j.m}$, for any $i, j, m \in N$.*

This definition is an alternative to the natural completely minimitive ranking function in definition 3, which defines r directly from propositions.

Definition 11 (Ranking Function r)**.** *Let $\mathscr{A}$ be a complete algebra over W, r is a natural negative ranking function iff r is a function from $\mathscr{A}$ into $N^+ = N \cup \infty$ such that:*

(i) $r(W) = 0$, $r(\varnothing) = \infty$;

(ii) if $A \in \mathscr{A}$, $min\{r(A), r(\bar{A})\} = 0$;

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(iii) for each countable $\mathcal{B} \subseteq \mathscr{A}$, $\bigcup \mathcal{B} \in \mathscr{A}$ *and* $r(\bigcup \mathcal{B}) = min\{r(B) \mid B \in \mathcal{B}\}$.

(iv) for each uncountable $\mathcal{B} \subseteq \mathscr{A}$, $\bigcup \mathcal{B} \in \mathscr{A}$ *and* $r(\bigcup \mathcal{B}) = min\{r(B) \mid B \in \mathcal{B}\}$.

r is called a natural completely minimitive ranking function.

Morevoer, a natural completely minimitive ranking function r can be reduced to some k according to Spohn (2012, p. 74):

> What about the issue of point versus set functions? If a ranking function is defined on a complete algebra and completely minimitive, it can be reduced to, or generated by, a point function on possibilities. If not, there is no guarantee. Huber(2006, pp.465f) gives an example in which a σ-minimitive ranking function k on an algebra is not reducible to a point function. But he does prove that, if the algebra is countably generated, then k is induced by a unique minimal point function (for details see Huber 2006, Theorem 1).

Further, I add two more definitions – a ranking pair $< k, r >$ and a ranking structure $<< W, \mathscr{A} >, < k, r >>$ as follows.

Definition 12 (Ranking Pair $< k, r >$)**.** *Let W be a set of possibilities and $\mathscr{A}$ be a complete algebra over W. k is an $\mathscr{A}$-measurable completely minimitive natural negative ranking function on W. r is a natural negative ranking function. Then $< k, r >$ is a ranking pair over $< W, \mathscr{A} >$.*

Definition 13 (Ranking Structure $<< W, \mathscr{A} >, < k, r >>$)**.** *Let W be a set of possibilities, $\mathscr{A}$ be a complete algebra over W and $< k, r >$ be the ranking pair. Then $<< W, \mathscr{A} >, < k, r >>$ is a ranking structure that coordinates $< W, \mathscr{A} >$ with natural numbers through a ranking pair $< k, r >$.*

The definitions and discussions above derive the following observation:

Observation 3. *Suppose $<< W_i, \mathscr{A}_j >, < k_{j.m}, r_{j.m} >>$ be any ranking structure, $i, j, m \in N$. $<< W_i, \mathscr{A}_j >, < k_{j.m}, r_{j.m} >>$ can be reduced to or induced by $<< W_i, k_{j.m} >, < \mathscr{A}_j, r_{j.m} >>$.*

Next, I provide an alternative definition which corresponds to $k(w \mid A)$ in definition 4.

Definition 14 (Conditionalization ck)**.** *Let $<< W, \mathscr{A} >, < k, r >>$ be a ranking structure and $A \in \mathscr{A}$ with $r(A), r(\bar{A}) < \infty$. Therefore, for any $w \in W$ the conditional negative rank of w given A is defined as*

$$ck(w, A) = \begin{cases} k(w) - r(A) & \textit{for} \quad w \in A, \\ \infty & \textit{for} \quad w \in \bar{A}. \end{cases}$$

Following from this definition and observation 2, ck is uniquely determined by a ranking structure $<< W, \mathscr{A} >, < k, r >>$. The next definition corresponds to $k(B \mid A)$ in definition 4.

Definition 15 (Conditionalization cr)**.** *Let $<< W, \mathscr{A} >, < k, r >>$ be a ranking structure and $A \in \mathscr{A}$ with $r(A), r(\bar{A}) < \infty$. For any $B \in \mathscr{A}$ the conditional negative rank of B given A is defined as*

$$cr(B, A) = min\{ck(w, A) \mid w \in B\} = r(A \cap B) - r(A).$$

The function $cr(B, A)$ is called the conditionalization of r by A.

Following from the definitions ck, cr and observation 2, cr is uniquely determined by a ranking structure $<< W, \mathscr{A} >, < k, r >>$.

Next, I add the following definitions for subsequent discussions.

Definition 16 (Pair Conditionalization $< ck, cr >$)**.** *Let $<< W, \mathscr{A} >, < k, r >>$ be a ranking structure. Then $< ck, cr >$ is a pair conditionalization over $<< W, \mathscr{A} >, < k, r >>$.*

Definition 17 (Pair Conditionalization Structure)**.** *Let $<< W, \mathscr{A} >, < k, r >>$ be a ranking structure and $< ck, cr >$ be a pair conditionalization. Then $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$ is a pair conditionalization structure over $<< W, \mathscr{A} >, < k, r >>$.*

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On the basis of the modified definitions, we can observe three mutually dependent structures: $< W, k, ck >$, $< \mathscr{A}, r, cr >$, and $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$. I summarize their connections in the following observation.

Observation 4. *Suppose $<< W_i, \mathscr{A}_j >, < k_{j.m}, r_{j.m} >, < ck_{j.m}, cr_{j.m} >>$ be any pair conditionalization structure, $i, j, m \in N$. Therefore, $<< W_i, \mathscr{A}_j >, < k_{j.m}, r_{j.m} >, < ck_{j.m}, cr_{j.m} >>$ can be reduced to, or induced by $<< W_i, k_{j.m}, ck_{j.m} >, < \mathscr{A}_j, r_{j.m}, cr_{j.m} >>$*

The proof can be derived from the definitions 9–17, and observation 3.

Further, the following two definitions – result-oriented conditionalization and the evidence-oriented conditionalization – will be defined on the basis of $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$.

Definition 18 (Result-Oriented Conditionalization)**.** *Let $<< W, \mathscr{A} >, < k, r >>$ be a ranking structure and $A \in \mathscr{A}$ such that $r(A), r(\bar{A}) < \infty$, and $n \in N^+$ Then the minimative result-oriented conditionalization ck_{result} is defined by*

$$ck_{result}(w, A, n) = \begin{cases} ck(w, A) & \textit{for } w \in A, \\ ck(w, \bar{A}) + n & \textit{for } w \in \bar{A}. \end{cases}$$

Definition 18 is an alternative to definition 5.

Definition 19 (Evidence-Oriented Conditionalization)**.** *Let $<< W, \mathscr{A} >, < k, r >>$ be a ranking structure and $A \in \mathscr{A}$ such that $r(A), r(\bar{A}) < \infty$, and $n \in N^+$. Then the minimative evidence-oriented conditionalization $ck_{evidence}(w, A, n)$ of k is defined by*

$$ck_{evidence}(w, A, n) = \begin{cases} k(w) - min\{r(A), n\} & \textit{for } w \in A, \\ k(w) + n - min\{r(A), n\} & \textit{for } w \in \bar{A}. \end{cases}$$

Definition 19 is an alternative to definition 6.

Definition 20 (Result-Oriented Conditionalization)**.** *Let $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$ be a pair conditionalization structure and $A \in \mathscr{A}$ such that $r(A), r(\bar{A}) < \infty$, then the result-oriented conditionalization cr_{result} maps from $< \mathscr{A}, \mathscr{A}, N^+ >$ to N^+, such that*
$cr_{result}(B, A, n) = min\{cr(B \setminus \bar{A}, A), cr(B \setminus A, \bar{A}) + n\}$.

Definition 20 is an alternative to definition 7.

Definition 21 (Evidence-Oriented Conditionalization $cr_{evidence}$)**.** *Let $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$ be a pair conditionalization structure and $A \in \mathscr{A}$ such that $r(A), r(\bar{A}) < \infty$, and $n \in N^+$. The evidence-oriented conditionalization $cr_{evidence}$ of r maps from $< \mathscr{A}, \mathscr{A}, N^+ >$ to N^+, such that*
$cr_{evidence}(B, A, n) = min\{cr(B \setminus \bar{A}, A) - min\{r(A), n\}, cr(\psi \setminus \phi, \bar{A}) + n - min\{r(A), n\}\}$.

Definition 21 is an alternative to definition 8. The last two definitions – definitions 20 and 21 – explicitly include the conditional ranks concerning the propositions that belong neither to A nor to $\bar{A}$. Besides, we can notice that a pair conditionalization structure $<< W, \mathscr{A} >, < k, r >, < ck, cr >>$ is a mutual base to define both cr_{result} and $cr_{evidence}$.

4 More Discussions

Now let's return to the two questions that are proposed in section 1.

To find an answer for the first question, we can use definitions 12–13 and observation 3. First of all, we cannot regard the domain of a ranking function as $W \cup \mathscr{A}$: the domain of a ranking pair $< k, r >$ is a pair $< W, \mathscr{A} >$, according to definition 12. Second, we can combine ranking functions and their domains together as a ranking structure $<< W, \mathscr{A} >, < k, r >>$ according to definition 13. Third, by observation 3, we see that a ranking structure $<< W, \mathscr{A} >, < k, r >>$ can be reduced to, or induced from $<< W, k >, < \mathscr{A}, r >>$. All these definitions and observations together provide an answer to the first question.

The answer to the second question are summarized as the following facts.

Fact 1. *Suppose $<< W, \mathscr{A} >, < k, r >>$ be a ranking structure. Then, $k^{-1}(n) \in \mathscr{A}$.*

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Proof. $k^{-1}(n) \in A$ collects all the possibilities w with $k(w) = n$. It can be seen as the $\bigcup r^{-1}(n)$, i.e. $k^{-1}(n) = \bigcup r^{-1}(n)$. According to (iv) and (v) in definition 1, for each $\mathcal{B} \subseteq \mathscr{A}$, $\bigcup \mathcal{B} \in \mathscr{A}$. Therefore, $k^{-1}(n) \in A$. $\square$

Fact 2. *Suppose* $<< W, \mathscr{A} >, < k, r >>$ *be a ranking structure. Then,* $k^{-1}(n) \in r^{-1}(n)$.

Proof. $\bigcup r^{-1}(n) \in \mathscr{A}$ implies that $\bigcup r^{-1}(n)$ is a proposition in $\mathscr{A}$. Therefore, $r(\bigcup r^{-1}(n)) = n$ because $r(\bigcup r^{-1}(n)) = min\{r(B) \mid B \in r^{-1}(n)\}$. $\square$

5 Conclusion

In the above sections, I propose some alternative definitions to clarify the ambiguity concerning the basic definitions in ranking theory. To generalize the basic structures contained in a ranking theory, I add some more new definitions for ranking theory, such as a ranking pair, a ranking structure and so forth. On the basis of these new definitions, I observe some connections between all these structures in section 4. In the end, the two questions (i) and (ii) made at the beginning are answered and proved on the basis of the new alternative definitions.

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